

# MEMOIR ON INTEGRATION OF ORDINARY DIFFERENTIAL EQUATIONS BY QUADRATURE<sup>1</sup>

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**Abstract.** The Riccati equations reducible to first-order linear equations by an appropriate change the dependent variable are singled out. All these equations are integrable by quadrature.

A wide class of linear ordinary differential equations reducible to algebraic equations is found. It depends on two arbitrary functions. The method for solving all these equations is given. The new class contains the constant coefficient equations and Euler's equations as particular cases.

*Keywords:* Linearizable Riccati equations, Higher-order linear equations reducible to algebraic equations, Generalization of Euler's equations.

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The old-fashioned title of the present paper indicates that it is dedicated to quite old topics. Namely, it deals with a problem on integration by quadrature of Riccati equations investigated, in terms of elementary functions, by Francesco Riccati and Daniel Bernoulli some 280 years ago for the special Riccati equations (see, e.g. [1])

$$y' = ay^2 + bx^\alpha, \quad a, b, \alpha = \text{const.},$$

and with an integration of higher-order linear equations by reducing them to algebraic equations. The later property was discovered by Leonard Euler in the 1740s for the constant coefficient equations

$$y^{(n)} + A_1 y^{(n-1)} + \dots + A_{n-1} y' + A_n y = 0, \quad A_1, \dots, A_n = \text{const.},$$

as well as for the equations of the form

$$x^n y^{(n)} + A_1 x^{n-1} y^{(n-1)} + \dots + A_{n-1} x y' + A_n y = 0, \quad A_1, \dots, A_n = \text{const.},$$

known as Euler's equations.

It will be shown in what follows that these classical results can be extended to wide classes of equations.

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# Chapter 1

## Riccati equation

### 1 Introduction

Consider the special Riccati equation

$$y' = ay^2 + bx^\alpha, \quad a, b, \alpha = \text{const.} \quad (1.1)$$

If  $\alpha = 0$ , Eq. (1.1) is integrable by the separation of variables:

$$\frac{dy}{ay^2 + b} = dx.$$

Another easily integrable case is  $\alpha = -2$ . Then the change of the dependent variable

$$z = \frac{1}{y}$$

maps Eq. (1.1) to the homogeneous equation

$$\frac{dz}{dx} = -\left[a - b\left(\frac{z}{x}\right)^2\right]$$

which is integrable by quadrature.

F. Riccati and D. Bernoulli noted independently that Eq. (1.1) can be transformed to the case  $\alpha = 0$ , and hence integrable by quadrature in terms of elementary functions if  $\alpha$  takes the values from the following two series:

$$\begin{aligned} \alpha &= -4, -\frac{8}{3}, -\frac{12}{5}, -\frac{16}{7}, \dots; \\ \alpha &= -\frac{4}{3}, -\frac{8}{5}, -\frac{12}{7}, -\frac{16}{9}, \dots \end{aligned} \quad (1.2)$$

The series (1.2) are given by the formula

$$\alpha = -\frac{4k}{2k-1} \quad \text{with} \quad k = \pm 1, \pm 2, \dots \quad (1.3)$$

It is manifest from (1.3) that both series in (1.2) have the limit  $\alpha = -2$ . For a derivation of the transformations mapping Eq. (1.1) with  $\alpha$  having the form (1.3) to an integrable form, see [1], Chapter 1, §6.

J. Liouville showed in 1841 that the solution to the special Riccati equation (1.1) is integrable by quadrature in terms of elementary functions only if  $\alpha$  has the form (1.3).

It is well known that the general Riccati equation

$$y' = P(x) + Q(x)y + R(x)y^2$$

can be rewritten as a linear second-order equation. But this kind of linearization by *raising the order* does not solve the integration problem. Therefore, I will investigate a possibility of linearization of the Riccati equations *without raising the order* and will show that all Riccati equations of this type can be integrated by quadrature.

## 2 The linearizable Riccati equations

The following theorem is closely related to the theory of nonlinear superpositions discussed in [2], Section 6.7.

**Theorem 2.1.** The first-order ordinary differential equation

$$y' = f(x, y) \quad (2.1)$$

can be reduced to a linear first-order equation

$$\frac{dz}{dx} = p(x) + q(x)z \quad (2.2)$$

by a change of the dependent variable  $y$ ,

$$z = z(y), \quad (2.3)$$

if and only if Eq. (2.1) can be written in the form

$$y' = T_1(x)\xi_1(y) + T_2(x)\xi_2(y) \quad (2.4)$$

such that the operators

$$X_1 = \xi_1(y)\frac{\partial}{\partial y}, \quad X_2 = \xi_2(y)\frac{\partial}{\partial y} \quad (2.5)$$

span a two-dimensional (or a one-dimensional if  $X_1$  and  $X_2$  are linearly dependent) Lie algebra called in [3] the VGL (Vessiot-Guldberg-Lie) algebra.

**Proof.** Let Eq. (2.1) be linearizable. Then we can assume that it is already reduced by a certain change of the dependent variable

$$z = \zeta(y) \quad (2.6)$$

to a linear equation (2.2),

$$\frac{dz}{dx} = p(x) + q(x)z.$$

The VGL algebra associated with Eq. (2.2) is two-dimensional and is spanned by the operators

$$\overline{X}_1 = \frac{\partial}{\partial z}, \quad \overline{X}_2 = z \frac{\partial}{\partial z}. \quad (2.7)$$

The form of Eq. (2.4) and the algebra property  $[X_1, X_2] = \alpha X_1 + \beta X_2$  remain unaltered under any change (2.3) of the dependent variable. Therefore, rewriting Eq. (2.2) in the original variable  $y = \zeta^{-1}(z)$  obtained by the inverse transformation to (2.6), we arrive at an equation of the form (2.4) for which the operators (2.5) span a two-dimensional Lie algebra. Since the equation obtained from Eq. (2.2) by the inverse transformation to (2.6) is the original equation (2.1) we have proved the “only if” part of the theorem.

Let us prove now the “if” part of the theorem. Namely, we have to demonstrate that any equation of the form (2.4) such that the operators (2.5) span a two-dimensional Lie algebra, is linearizable. If the operators (2.5) are linearly dependent, then  $\xi_2(x) = \gamma \xi_1(x)$ ,  $\gamma = \text{const.}$ , and hence Eq. (2.4) has the form

$$y' = [T_1(x) + \gamma T_2(x)] \xi_1(y).$$

It can be reduced to the linear equation

$$z' = T_1(x) + \gamma T_2(x).$$

upon introducing a canonical variable  $z$  for

$$X_1 = \xi_1(y) \frac{\partial}{\partial y}$$

by solving the equation  $X_1(z) = 1$ .

Suppose now that the operators (2.5) are linearly independent. It is clear that Eq. (2.9) will be linearized if one transforms the operators (2.5) to the form (2.7). One can assume that the first operator (2.5) has been

already written, in a proper variable  $z$ , in the form of the first operator  $\overline{X}_1$  given in (2.7):

$$X_1 = \frac{\partial}{\partial z}.$$

Let the second operator (2.5) be written in the variable  $z$  as follows:

$$X_2 = f(z) \frac{\partial}{\partial z}.$$

We have

$$[X_1, X_2] = f'(z) \frac{\partial}{\partial z}$$

and the requirement  $[X_1, X_2] = \alpha X_1 + \beta X_2$  that  $X_1, X_2$  span a Lie algebra  $L_2$  yields the differential equation

$$f' = \alpha + \beta f,$$

where not both  $\alpha$  and  $\beta$  vanish because otherwise the operators  $X_1$  and  $X_2$  will be linearly dependent. hence  $f'(x) \neq 0$ . Solving the above differential equation, we obtain

$$\begin{aligned} f &= \alpha z + C \quad \text{if } \beta = 0, \\ f &= C e^{\beta z} - \frac{\alpha}{\beta} \quad \text{if } \beta \neq 0. \end{aligned}$$

$$\begin{aligned} f = ax + C &\implies X_2 = ax \frac{d}{dx} + CX_1, \quad \text{if } b = 0, \\ f = C e^{bx} - \frac{a}{b} &\implies X_2 = C e^{bx} \frac{d}{dx} - \frac{a}{b} X_1, \quad \text{if } b \neq 0. \end{aligned}$$

In the first case we have

$$X_2 = \alpha z \frac{\partial}{\partial z} + CX_1,$$

and hence a basis of  $L_2$  is provided by (2.7). In the second case we have

$$X_2 = C e^{\beta z} \frac{\partial}{\partial z} - \frac{\alpha}{\beta} X_1,$$

and one can take basis operators, by assigning  $\beta z$  as new  $z$ , in the form

$$X_1 = \frac{\partial}{\partial z}, \quad X_2 = e^z \frac{\partial}{\partial z}.$$

Finally, substituting  $\bar{z} = e^{-z}$  we arrive at the basis (2.7), thus completing the proof of the theorem.

The following theorem characterizes all Riccati equations that can be reduced to first-order linear equations by changing the dependent variable (see [4], Russian ed., Theorem 4.3; [3], Section 11.2.5 and Note [11.4]; see also Theorem 3.2.2 in [2]).

**Theorem 2.2.** The following two conditions 1° and 2° are equivalent and provide the necessary and sufficient conditions for the Riccati equation

$$y' = P(x) + Q(x)y + R(x)y^2 \quad (2.8)$$

to be linearizable by a change of the dependent variable (2.3),  $z = z(y)$ .

1°. Eq. (2.8) has a constant solution  $y = c$  including  $c = \infty$ .

2°. Eq. (2.8) has either the form

$$y' = Q(x)y + R(x)y^2 \quad (2.9)$$

with any functions  $Q(x)$  and  $R(x)$ , or the form

$$y' = P(x) + Q(x)y + k[Q(x) - kP(x)]y^2 \quad (2.10)$$

with any functions  $P(x)$ ,  $Q(x)$  and any constant  $k$ .

**Proof.** The conditions 1° and 2° are equivalent. Indeed, let Eq. (2.8) have a constant solution  $y = c$ . Then

$$0 = P(x) + Q(x)c + R(x)c^2. \quad (2.11)$$

If  $c = 0$ , Eq. (2.11) yields  $P(x) = 0$ , and hence Eq. (2.8) has the form (2.9).

If  $c \neq 0$ , Eq. (2.11) yields

$$R(x) = -\frac{1}{c^2}[P(x) + cQ(x)] = -\frac{1}{c}[Q(x) + \frac{1}{c}P(x)].$$

Denoting  $k = -1/c$  we get

$$R(x) = k[Q(x) - kP(x)].$$

Hence Eq. (2.8) has the form (2.10). Thus, we have proved that  $1^\circ \Rightarrow 2^\circ$ .

Conversely, let Eq. (2.8) satisfy the condition 2°. It is manifest that Eq. (2.9) has the constant solution  $y = 0$ . Furthermore, one can verify that Eq. (2.10) has the constant solution  $y = -1/k$ . This proves that  $2^\circ \Rightarrow 1^\circ$ .

Let us turn to the necessary and sufficient conditions for linearization. The operators

$$X_1 = y \frac{\partial}{\partial y}, \quad X_2 = y^2 \frac{\partial}{\partial y}$$

associated with Eq. (2.9) have the commutator  $[X_1, X_2] = X_2$ , and hence span a two-dimensional Lie algebra. Furthermore, the operators

$$X_1 = (1 - k^2 y^2) \frac{\partial}{\partial y}, \quad X_2 = (y + ky^2) \frac{\partial}{\partial y}$$

associated with Eq. (2.10) have the commutator  $[X_1, X_2] = X_1 + 2kX_2$ . Hence, they also span a two-dimensional Lie algebra. Therefore, according to Theorem 2.1, the condition 2° is sufficient for linearization.

Note that the equation  $y' = P(x) + Q(x)y$  can be regarded as a particular case of Eq. (2.10) with  $k = 0$ . Since Eq. (2.10) has the constant solution  $y = -1/k$ , we conclude that the linear equation  $y' = P(x) + Q(x)y$  has the constant solution  $y = \infty$ . We conclude that any linearizable equation has a constant solution because the change of the dependent variable  $z = z(y)$  maps a constant solution into a constant solution of the transformed equation. Hence, the condition 1° is necessary for linearization. This completes the proof of the theorem due to the equivalence of the conditions 1° and 2°.

Now we will use Theorem 2.2 for integrating the linearizable Riccati equations. We will find linearizing transformations for the equations (2.9) and (2.10). We will assume that  $k$  in Eq. (2.10) is a real number.

### 3 Linearization and integration of Eq. (2.9)

Invoking Eqs. (6.7.5), (6.7.6) from [2], replacing there  $t$  and  $x^i$  by  $x$  and  $y$ , respectively, and identifying  $T_1(t)$  and  $T_2(t)$  with  $Q(x)$  and  $R(x)$ , respectively, we see that the VGL (Vessiot-Guldberg-Lie) algebra associated with Eq. (2.9) is a two-dimensional algebra  $L_2$  spanned by the operators

$$X_1 = y \frac{\partial}{\partial y}, \quad X_2 = y^2 \frac{\partial}{\partial y}.$$

Their commutator is  $[X_1, X_2] = X_2$ . Introducing the new basis

$$X'_1 = X_2, \quad X'_2 = -X_1$$

i.e. taking

$$X'_1 = y^2 \frac{\partial}{\partial y}, \quad X'_2 = -y \frac{\partial}{\partial y} \tag{3.1}$$

we have a basis in  $L_2$  satisfying the commutator relation

$$[X'_1, X'_2] = X'_1. \tag{3.2}$$

Let us find a change of the dependent variable,  $z = z(y)$ , such that Eq. (2.9) becomes a linear equation (2.2),

$$\frac{dz}{dx} = p(x) + q(x)z.$$

The VGL algebra of this equation is spanned by the operators (2.7),

$$\overline{X}_1 = \frac{\partial}{\partial z}, \quad \overline{X}_2 = z \frac{\partial}{\partial z}$$

whose commutator has the same form as Eq. (3.2), i.e.  $[\overline{X}_1, \overline{X}_2] = \overline{X}_1$ . Consequently, the linearizing transformation  $z = z(y)$  is determined by the equations  $X'_1(z) = 1$ ,  $X'_2(z) = z$ , or

$$y^2 \frac{dz}{dy} = 1, \quad -y \frac{dz}{dy} = z. \quad (3.3)$$

Integrating the first equation (3.3) we obtain

$$z = -\frac{1}{y} + A.$$

Substituting in the second equation (3.3) we get  $A = 0$ . Thus, the linearizing transformation is

$$z = -\frac{1}{y}. \quad (3.4)$$

In this variable, Eq. (2.9) becomes the linear equation

$$z' = R(x) - Q(x)z, \quad (3.5)$$

whence

$$z = \left[ C + \int R(x) e^{\int Q(x) dx} dx \right] e^{-\int Q(x) dx}, \quad C = \text{const.}$$

Substituting in Eq. (3.4) we finally arrive at the solution to Eq. (2.9):

$$y = - \left[ C + \int R(x) e^{\int Q(x) dx} dx \right]^{-1} e^{\int Q(x) dx}. \quad (3.6)$$

**Remark 3.1.** In Section 12.1 the solution (3.6) is obtained by an alternative method.

**Example 3.1.** Consider the equation

$$y' = \frac{y}{x} + R(x)y^2.$$

By Eq. (3.6) yields its solution

$$y = \frac{x}{C - \int xR(x)dx}.$$

**Example 3.2.** The equation

$$y' = \frac{y}{x} + 3xy^2$$

is a particular case of the previous equation. Its solution is

$$y = \frac{x}{C - x^2}.$$

**Example 3.3.** The equation

$$y' = y + \frac{1}{x}y^2$$

has the solution

$$y = \frac{e^x}{C + \text{Ei}(x)}, \quad \text{where} \quad \text{Ei}(x) = - \int_{-\infty}^x \frac{e^t}{t} dt.$$

## 4 Linearization and integration of Eq. (2.10)

Now we identify the coefficients  $P(x)$  and  $Q(x)$  of Eq. (2.10),

$$y' = P(x) + Q(x)y + k[Q(x) - kP(x)]y^2, \quad (2.10)$$

with the coefficients  $T_1(t)$  and  $T_2(t)$  of Eqs. (6.7.5), (6.7.6) from [2] and associated with Eq. (2.10) the VGL algebra spanned by the operators

$$X_1 = (1 - k^2y^2)\frac{\partial}{\partial y}, \quad X_2 = (y + ky^2)\frac{\partial}{\partial y}.$$

Their commutator is  $[X_1, X_2] = X_1 + 2kX_2$ . We assume that  $k \neq 0$ , since otherwise Eq. (2.10) is already linear. Therefore, setting

$$X'_1 = X_1 + 2kX_2, \quad X'_2 = -X_1/(2k)$$

we obtain the new basis

$$X'_1 = (1 + ky)^2 \frac{\partial}{\partial y}, \quad X'_2 = \frac{1}{2k} (k^2 y^2 - 1) \frac{\partial}{\partial y}, \quad (4.1)$$

satisfying the commutator relation (3.2),  $[X'_1, X'_2] = X'_1$ .

The transformation  $z = z(y)$  of the operators (4.1) to the form (2.7) is determined by the equations

$$X'_1(z) = 1, \quad X'_2(z) = z,$$

or

$$(1 + ky)^2 \frac{dz}{dy} = 1, \quad \frac{1}{2k} (k^2 y^2 - 1) \frac{dz}{dy} = z. \quad (4.2)$$

Integrating the first equation (4.2) we obtain

$$z = A - \frac{1}{k(1 + ky)}.$$

Substituting in the second equation (4.2) we get  $A = 1/(2k)$ . Thus, the linearizing transformation is

$$z = \frac{ky - 1}{2k(ky + 1)}. \quad (4.3)$$

Eq. (4.3) yields:

$$y = \frac{1 + 2kz}{k(1 - 2kz)}. \quad (4.4)$$

Substituting (4.4) in Eq. (2.10) we arrive at the linear equation

$$z' = \frac{1}{2k} Q(x) + [Q(x) - 2kP(x)]z, \quad (4.5)$$

whence

$$z = \frac{1}{2k} \left[ C + \int Q(x) e^{\int [2kP(x) - Q(x)] dx} dx \right] e^{\int [Q(x) - 2kP(x)] dx}. \quad (4.6)$$

Substituting (4.6) in (4.4) we will obtain the solution to Eq. (2.10).

**Remark 4.1.** We have assumed that  $Q(x) \neq 0$  because otherwise Eq. (2.10) is separable,

$$y' = P(x)(1 - k^2 y^2).$$

**Remark 4.2.** Eq. (2.10) has the constant solution

$$y = -\frac{1}{k}. \quad (4.7)$$

We excluded the singular solution (4.7) in the above calculations and assumed that  $1 + ky \neq 0$ .

**Example 4.1.** Let us integrate the equation

$$y' = x + 2xy + xy^2. \quad (4.8)$$

Here

$$P(x) = x, \quad Q(x) = 2x, \quad k = 1.$$

Therefore the linearized equation (4.5) is written  $z' = x$ . Integrating it and denoting the constant of integration by  $C/2$ , we obtain

$$z = \frac{1}{2}(C + x^2)$$

in accordance with Eq. (4.6). Substituting the above  $z$  in Eq. (4.4) we obtain the following solution to Eq. (4.8):

$$y = \frac{1 + C + x^2}{1 - C - x^2}. \quad (4.9)$$

Remark 4.2 gives also the singular solution  $y = -1$ .

**Remark 4.3.** Eq. (4.8) can also be integrated by separating the variables.

**Example 4.2.** The equation

$$y' = x^2 + (x + x^2)y + \frac{1}{4}(2x + x^2)y^2 \quad (4.10)$$

has the form (2.10) with

$$P(x) = x^2, \quad Q(x) = x + x^2, \quad k = \frac{1}{2}.$$

The linearized equation (4.5) is written

$$z' = x + x^2 + xz$$

and has the solution

$$z = \left( C + \int (x + x^2)e^{-x^2/2} dx \right) e^{x^2/2}.$$

Substitution in Eq. (4.4) yields

$$y = 2 \frac{1+z}{1-z}.$$

Hence, the solution to Eq. (4.10) is given by

$$y = 2 \frac{1 + \left( C + \int (x + x^2) e^{-x^2/2} dx \right) e^{x^2/2}}{1 - \left( C + \int (x + x^2) e^{-x^2/2} dx \right) e^{x^2/2}}. \quad (4.11)$$

## Chapter 2

# Higher-order linear equations reducible to algebraic equation

### 5 Introduction

The linear ordinary differential equations with constant coefficients

$$y^{(n)} + A_1 y^{(n-1)} + \cdots + A_{n-1} y' + A_n y = 0, \quad A_1, \dots, A_n = \text{const.}, \quad (5.1)$$

and Euler's equations

$$x^n y^{(n)} + A_1 x^{n-1} y^{(n-1)} + \cdots + A_{n-1} x y' + A_n y = 0, \quad A_1, \dots, A_n = \text{const.}, \quad (5.2)$$

are discussed practically in all textbooks on differential equations. They are useful in applications. The most remarkable property is that both equations (5.1) and (5.2) are *reducible to algebraic equations*. Namely, their fundamental systems of solutions, and hence the general solutions can be obtained by solving algebraic equations. Then one can integrate the corresponding non-homogeneous linear equations by using the method of variation of parameters. It is significant to understand the nature of the reducibility and to extend the class of linear equations reducible to algebraic equations.

In the present paper we will investigate this problem and find a wide class of linear ordinary differential equations that are reducible to algebraic equations. The new class depends on two arbitrary functions of  $x$  and contains the equations (5.1) and (5.2) as particular cases. The method for solving all these equations is given in Sections 8 and 10, and illustrated by examples in Section 9.

The main statement for the second-order equations is as follows (Section 8).

**Theorem 5.1.** The linear second-order equations

$$P(x)y'' + Q(x)y' + R(x)y = F(x)$$

whose general solution can be obtained by solving algebraic equations and by quadratures, have the form

$$\phi^2 y'' + (A + \phi' - 2\sigma)\phi y' + (B - A\sigma + \sigma^2 - \phi\sigma')y = F(x), \quad (5.3)$$

where  $\phi = \phi(x)$ ,  $\sigma = \sigma(x)$  and  $F(x)$  are arbitrary (smooth) functions, and  $A, B = \text{const.}$  The homogeneous equation (5.3),

$$\phi^2 y'' + (A + \phi' - 2\sigma)\phi y' + (B - A\sigma + \sigma^2 - \phi\sigma')y = 0, \quad (5.4)$$

has the solutions of the form

$$y = e^{\int \frac{\sigma(x) + \lambda}{\phi(x)} dx}, \quad (5.5)$$

where  $\lambda$  satisfies the *characteristic equation*

$$\lambda^2 + A\lambda + B = 0. \quad (5.6)$$

If the characteristic equation (5.6) has distinct real roots  $\lambda_1 \neq \lambda_2$ , the general solution to Eq. (5.4) is given by

$$y(x) = K_1 e^{\int \frac{\sigma(x) + \lambda_1}{\phi(x)} dx} + K_2 e^{\int \frac{\sigma(x) + \lambda_2}{\phi(x)} dx}, \quad K_1, K_2 = \text{const.} \quad (5.7)$$

In the case of complex roots,  $\lambda_1 = \gamma + i\theta$ ,  $\lambda_2 = \gamma - i\theta$ , the general solution to Eq. (5.4) is given by

$$y(x) = \left[ K_1 \cos \left( \theta \int \frac{dx}{\phi(x)} \right) + K_2 \sin \left( \theta \int \frac{dx}{\phi(x)} \right) \right] e^{\int \frac{\sigma(x) + \gamma}{\phi(x)} dx}. \quad (5.8)$$

If the characteristic equation (5.6) has equal roots  $\lambda_1 = \lambda_2$ , the general solution to Eq. (5.4) is given by

$$y = \left[ K_1 + K_2 \int \frac{dx}{\phi(x)} \right] e^{\int \frac{\sigma(x) + \lambda_1}{\phi(x)} dx}, \quad K_1, K_2 = \text{const.} \quad (5.9)$$

The general solution of the non-homogeneous equation (5.3) can be obtained by the method of variation of parameters.

**Remark 5.1.** The equations with constant coefficients and Euler's equation are the simplest representatives of Eqs. (5.4). Namely, setting  $\phi(x) = 1$ ,  $\sigma(x) = 0$  we obtain the second-order equation with constant coefficients

$$y'' + Ay' + By = 0,$$

and Eq. (5.5) yields the well-known formula

$$y = e^{\lambda x},$$

where  $\lambda$  is determined by the characteristic equation (5.5).

If we set  $\phi(x) = x$ ,  $\sigma(x) = 0$ , we obtain the second-order Euler's equation (5.2) written in the form

$$x^2 y'' + (A + 1)xy' + By = 0.$$

Then Eq. (5.5) yields the particular solutions for Euler's equations:

$$y = x^\lambda,$$

where  $\lambda$  is determined again by the characteristic equation (5.5). For details, see Section 9. For other functions  $\phi(x)$  and  $\sigma(x) = 0$ , Eqs. (5.5) are new.

## 6 Constant coefficient and Euler's equations from the group standpoint

For the sake of simplicity, we will consider in this section second-order equations

$$y'' + f(x)y' + g(x)y = 0. \quad (6.1)$$

### 6.1 Equations with constant coefficients

Let us begin with the equations with constant coefficients

$$y'' + Ay' + By = 0, \quad A, B = \text{const.} \quad (6.2)$$

Eq. (6.2) is invariant under the one-parameter groups of translations in  $x$  and dilations in  $y$ , since it does not involve the independent variable  $x$  explicitly (the coefficients  $A$  and  $B$  are constant) and is homogeneous in the dependent variable  $y$ . In other words, Eq. (6.2) admits the generators

$$X_1 = \frac{\partial}{\partial x}, \quad X_2 = y \frac{\partial}{\partial y} \quad (6.3)$$

of the translations in  $x$  and dilations in  $y$ . We use them as follows [3].

Let us find the invariant solution for  $X = X_1 + \lambda X_2$ , i.e.

$$X = \frac{\partial}{\partial x} + \lambda y \frac{\partial}{\partial y} \quad \lambda = \text{const.} \quad (6.4)$$

The characteristic equation

$$\frac{dy}{y} = \lambda \frac{dx}{x}$$

of the equation

$$X(J) \equiv \frac{\partial J}{\partial x} + \lambda y \frac{\partial J}{\partial y} = 0$$

for the invariants  $J(x, y)$  yields one functionally independent invariant

$$J = y e^{-\lambda x}.$$

According to the general theory, the invariant solution is given by  $J = C$  with an arbitrary constant  $C$ . Thus, the general form of the invariant solutions for the operator (6.4) is

$$y = C e^{\lambda x}, \quad C = \text{const.}$$

Since Eq. (6.2) is homogeneous one can set  $C = 1$  and obtain *Euler's substitution*:

$$y = e^{\lambda x}. \quad (6.5)$$

As well known, the substitution (6.5) reduces Eq. (6.2) to the quadratic equation (*characteristic equation*)

$$\lambda^2 + A\lambda + B = 0. \quad (6.6)$$

If Eq. (6.6) has two distinct roots,  $\lambda_1 \neq \lambda_2$ , then Eq. (6.5) provides two linearly independent solutions

$$y_1 = e^{\lambda_1 x}, \quad y_2 = e^{\lambda_2 x},$$

and hence, a fundamental set of solutions. If the roots are real, the general solution to Eq. (6.2) is

$$y(x) = K_1 e^{\lambda_1 x} + K_2 e^{\lambda_2 x}, \quad K_1, K_2 = \text{const.} \quad (6.7)$$

If the roots are complex,  $\lambda_1 = \gamma + i\theta$ ,  $\lambda_2 = \gamma - i\theta$ , the general solution to Eq. (6.2) is given by

$$y(x) = [K_1 \cos(\theta x) + K_2 \sin(\theta x)] e^{\gamma x}, \quad K_1, K_2 = \text{const.} \quad (6.8)$$

In the case of equal roots  $\lambda_1 = \lambda_2$ , standard texts in differential equations make a guess, without motivation, that the general solution has the form

$$y(x) = (K_1 + K_2 x) e^{\lambda_1 x}, \quad K_1, K_2 = \text{const.} \quad (6.9)$$

The motivation is given in [3], Section 13.2.2, and states the following.

**Lemma 6.1.** Eq. (6.2) can be mapped to the equation  $z'' = 0$  by a linear change of the dependent variable

$$y = \sigma(x) z, \quad \sigma(x) \neq 0, \quad (6.10)$$

if and only if the characteristic equation (6.6) has equal roots. Specifically, if  $\lambda_1 = \lambda_2$ , the substitution

$$y = z e^{\lambda_1 x} \quad (6.11)$$

carries Eq. (6.2) to the equation  $z'' = 0$ . Substituting in (6.11) the solution  $z = K_1 + K_2 x$  of the equation  $z'' = 0$ , we obtain the general solution (6.9) to Eq. (6.2) whose coefficients satisfy the condition of equal roots for Eq. (6.6):

$$A^2 - 4B = 0. \quad (6.12)$$

**Proof.** The reckoning shows that after the substitution (6.10) Eq. (6.1) becomes (see, e.g. [2], Section 3.3.2)

$$z'' + I(x)z = 0,$$

where

$$I(x) = g(x) - \frac{1}{4} f^2(x) - \frac{1}{2} f'(x)$$

and the function  $\sigma(x)$  in the transformation (6.10) has the form

$$\sigma(x) = e^{-\frac{1}{2} \int f(x) dx}. \quad (6.13)$$

Hence, Eq. (6.1) is carried into equation  $z'' = 0$  if and only if  $I(x) = 0$ , i.e.

$$f^2(x) + 2f'(x) - 4g(x) = 0. \quad (6.14)$$

In the case of Eq. (6.2), the condition (6.14) is identical with Eq. (6.12), and the function  $\sigma(x)$  given by Eq. (6.13) becomes

$$\sigma(x) = e^{-\frac{A}{2} x}. \quad (6.15)$$

Furthermore, under the condition Eq. (6.12), the repeated root of Eq. (6.6) is  $\lambda_1 = -A/2$ . Therefore Eq. (6.15) can be written

$$\sigma(x) = e^{\lambda_1 x}$$

and we arrive at the substitution (6.11), and hence at the solution (6.9), thus proving the lemma.

## 6.2 Euler's equation

Consider Euler's equation

$$x^2 y'' + A x y' + B y = 0, \quad A, B = \text{const.} \quad (6.16)$$

It is double homogeneous (see [2], Section 6.6.1), i.e. admits the dilation groups in  $x$  and in  $y$  with the generators

$$X_1 = x \frac{\partial}{\partial x}, \quad X_2 = y \frac{\partial}{\partial y}. \quad (6.17)$$

We proceed as in Section 6.1 and find the invariant solutions for the linear combination  $X = X_1 + \lambda X_2$ :

$$X = x \frac{\partial}{\partial x} + \lambda y \frac{\partial}{\partial y}, \quad \lambda = \text{const.} \quad (6.18)$$

The characteristic equation

$$\frac{dy}{y} = \lambda \frac{dx}{x}$$

of the equation  $X(J) = 0$  for the invariants  $J(x, y)$  yields the invariant

$$J = y x^{-\lambda}$$

for the operator (6.18). The invariant solutions are given by  $J = C$ , whence

$$y = C x^\lambda, \quad C = \text{const.}$$

Due to the homogeneity of Eq. (6.2) we can set  $C = 1$  and obtain

$$y = x^\lambda. \quad (6.19)$$

Differentiating and multiplying by  $x$ , we have:

$$x y' = \lambda x^\lambda, \quad x^2 y'' = \lambda(\lambda - 1) x^\lambda.$$

Substituting in Eq. (6.18) and dividing by the common factor  $C x^\lambda$  we obtain the following *characteristic equation* for Euler's equation (6.16):

$$\lambda^2 + (A - 1) \lambda + B = 0. \quad (6.20)$$

**Remark 6.1.** According to Eqs. (6.6), (6.20), the characteristic equation for Euler's equation written in the form

$$x^2 y'' + (A + 1) x y' + B y = 0 \quad (6.21)$$

is identical with the characteristic equation (6.6) for Eq. (6.2) with constant coefficients.

## 7 New examples of reducible equations

### 7.1 First example

Let us find the linear second-order equations (6.1),

$$y'' + f(x)y' + g(x)y = 0, \quad (6.1)$$

admitting the operator

$$X_1 = x^\alpha \frac{\partial}{\partial x}, \quad (7.1)$$

where  $\alpha$  is any real-valued parameter. Taking the second prolongation of  $X_1$ ,

$$X_1 = x^\alpha \frac{\partial}{\partial x} - \alpha x^{\alpha-1} y' \frac{\partial}{\partial y'} - [\alpha(\alpha-1)x^{\alpha-2}y' + 2x^{\alpha-1}y''] \frac{\partial}{\partial y''},$$

we write the invariance condition of Eq. (6.1),

$$X_1(y'' + f(x)y' + g(x)y) \Big|_{(6.1)} = 0,$$

and obtain:

$$x^{\alpha-2}[x^2f' + \alpha xf - \alpha(\alpha-1)]y' + x^{\alpha-1}[xg' + 2\alpha g]y = 0. \quad (7.2)$$

Since Eq. (7.2) should be satisfied identically in the variables  $x, y, y'$ , it splits into two equations:

$$x^2f' + \alpha xf - \alpha(\alpha-1) = 0, \quad xg' + 2\alpha g = 0. \quad (7.3)$$

Solving the first-order linear differential equations (7.3) for the unknown functions  $f(x)$  and  $g(x)$ , we obtain:

$$f(x) = \frac{\alpha}{x} + Ax^{-\alpha}, \quad g(x) = Bx^{-2\alpha}, \quad A, B = \text{const.}$$

Thus, we arrive at the following linear equation admitting the operator (7.1):

$$x^{2\alpha}y'' + (Ax^\alpha + \alpha x^{2\alpha-1})y' + By = 0, \quad A, B = \text{const.} \quad (7.4)$$

**Remark 7.1.** When  $\alpha = 0$ , Eq. (7.4) yields the equation (6.2) with constant coefficients. When  $\alpha = 1$ , Eq. (7.4), upon setting  $A + 1$  as a new coefficient  $A$ , coincides with Euler's equation (6.16).

Since Eq. (7.4) is linear homogeneous, it admits, along with (7.1), the operator

$$X_2 = y \frac{\partial}{\partial y}.$$

Now we proceed as in Section 6 and find the invariant solutions for the linear combination  $X = X_1 + \lambda X_2$ :

$$X = x^\alpha \frac{\partial}{\partial x} + \lambda y \frac{\partial}{\partial y}, \quad \lambda = \text{const.}$$

Let  $\alpha \neq 1$ . The characteristic equation

$$\frac{dy}{y} = \lambda \frac{dx}{x^\alpha}$$

of the equation  $X(J) = 0$  for the invariants  $J(x, y)$  yields the invariant

$$J = y e^{\frac{\lambda}{\alpha-1} x^{1-\alpha}}.$$

Hence, the invariant solutions are obtained by setting  $J = C$ , whence letting  $C = 1$  we have

$$y = e^{\frac{\lambda}{1-\alpha} x^{1-\alpha}}. \quad (7.5)$$

Differentiating we have:

$$y' = \lambda x^{-\alpha} e^{\frac{\lambda}{1-\alpha} x^{1-\alpha}}, \quad y'' = [\lambda^2 x^{-2\alpha} - \lambda \alpha x^{-\alpha-1}] e^{\frac{\lambda}{1-\alpha} x^{1-\alpha}}.$$

Substituting in Eq. (7.4) and dividing by the non-vanishing factor  $e^{\frac{\lambda}{1-\alpha} x^{1-\alpha}}$ , we obtain the following *characteristic equation* for Eq. (7.4):

$$\lambda^2 + A \lambda + B = 0. \quad (7.6)$$

Eq. (7.6) is identical with the characteristic equation (6.6) for the equation (6.2) with constant coefficients.

**Remark 7.2.** Eq. (7.5) contains Euler's substitution (6.5) for the equation (6.2) with constant coefficients as a particular case  $\alpha = 0$ .

**Remark 7.3.** Using the statement that Eq. (6.1) is mapped to the equation  $z'' = 0$  if and only if the function

$$I(x) = g(x) - \frac{1}{4} f^2(x) - \frac{1}{2} f'(x)$$

vanishes (see Lemma 6.1), one can verify that Eq. (7.5) is equivalent by function to the equation  $z'' = 0$  if and only if

$$A^2 - 4B = 0 \quad \text{and} \quad \alpha = 0 \text{ or } \alpha = 2. \quad (7.7)$$

The first equation in (7.7) means that the characteristic equation (7.6) has a repeated root, and hence there is only one solution of the form (7.5). Then, using the reasoning of Lemma 6.1 we can show that the general solution to Eq. (7.4) with  $A^2 - 4B = 0$ ,  $\alpha = 2$ , i.e. of the equation

$$y'' + \left( \frac{A}{x^2} + \frac{2}{x} \right) y' + \frac{B}{x^4} y = 0, \quad A^2 - 4B = 0,$$

is given by

$$y = \left( K_1 + \frac{K_2}{x} \right) e^{-\frac{\lambda}{x}}, \quad (7.8)$$

where  $K_1, K_2$  are arbitrary constants and  $\lambda$  is the repeated root of the characteristic equation (7.6). For a more general statement, see Section 9.

## 7.2 Second example

Let us find the linear second-order equations (6.1) admitting the projective group with the generator

$$X_1 = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}. \quad (7.9)$$

Taking the second prolongation of the operator (7.9),

$$X_1 = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y} + (y - xy') \frac{\partial}{\partial y'} - 3xy'' \frac{\partial}{\partial y''},$$

and writing the invariance condition of Eq. (6.1),

$$X_1(y'' + f(x)y' + g(x)y) \Big|_{(6.1)} = 0,$$

we obtain:

$$x(xf' + 2f)y' + (x^2g' + 4xg + f)y = 0. \quad (7.10)$$

Eqs. (7.10) yield:

$$f(x) = \frac{A}{x^2}, \quad g(x) = \frac{B}{x^4} - \frac{A}{x^3}, \quad A, B = \text{const.}$$

Thus, we arrive at the following linear equation admitting the operator (7.9):

$$x^4 y'' + Ax^2 y' + (B - Ax)y = 0, \quad A, B = \text{const.} \quad (7.11)$$

Eq. (7.11) is linear homogeneous, and hence admits, along with (7.9), the operator

$$X_2 = y \frac{\partial}{\partial y}.$$

Now we proceed as in Section 6 and find the invariant solutions for the linear combination  $X = X_1 + \lambda X_2$ :

$$X = x^2 \frac{\partial}{\partial x} + (x + \lambda)y \frac{\partial}{\partial y}, \quad \lambda = \text{const.}$$

Rewriting the characteristic equation of the equation  $X(J) = 0$  for the invariants  $J(x, y)$  in the form

$$\frac{dy}{y} = \frac{x + \lambda}{x^2} dx$$

we obtain the invariant

$$J = \frac{y}{x} e^{\frac{\lambda}{x}}.$$

Setting  $J = C$  and letting  $C = 1$  we obtain the following form of the invariant solutions:

$$y = x e^{-\frac{\lambda}{x}}. \quad (7.12)$$

Substituting (7.12) in Eq. (7.11) we reduce the differential equation (7.11) to the algebraic equation

$$\lambda^2 + A\lambda + B = 0$$

which is identical with the characteristic equation (6.6) for the equation (6.2) with constant coefficients.

**Example 7.1.** Solve the equation

$$y'' + \frac{\omega^2}{x^4} y = 0, \quad \omega = \text{const.} \quad (7.13)$$

This is an equation of the form (7.11) with  $A = 0$ ,  $B = \omega^2$ . The algebraic equation (6.6) yields  $\lambda_1 = -i\omega$ ,  $\lambda_2 = i\omega$ , and hence we have two independent invariant solutions (7.12):

$$y_1 = x e^{i\frac{\omega}{x}}, \quad y_2 = x e^{-i\frac{\omega}{x}}.$$

Taking their real and imaginary parts, just like in the case of constant coefficient equations, we obtain the following fundamental system of solutions:

$$y_1 = x \cos\left(\frac{\omega}{x}\right), \quad y_2 = x \sin\left(\frac{\omega}{x}\right). \quad (7.14)$$

Hence, the general solution to Eq. (7.12) is given by

$$y = x \left[ C_1 \cos\left(\frac{\omega}{x}\right) + C_2 \sin\left(\frac{\omega}{x}\right) \right].$$

We can also solve the non-homogeneous equation

$$y'' + \frac{\omega^2}{x^4} y = F(x), \quad (7.15)$$

e.g. by the method of *variation of parameters*, and obtain

$$\begin{aligned} y = x & \left[ C_1 \cos\left(\frac{\omega}{x}\right) + C_2 \sin\left(\frac{\omega}{x}\right) \right] \\ & + \frac{x}{\omega} \left[ \cos\left(\frac{\omega}{x}\right) \int x F(x) \sin\left(\frac{\omega}{x}\right) dx - \sin\left(\frac{\omega}{x}\right) \int x F(x) \cos\left(\frac{\omega}{x}\right) dx \right]. \end{aligned} \quad (7.16)$$

## 8 General result for second-order equations

### 8.1 Main statements

Note that the operators given in Eqs. (6.3), (6.17), (7.1), (7.9) are particular cases of the generator

$$X_1 = \phi(x) \frac{\partial}{\partial x} + \sigma(x) y \frac{\partial}{\partial y} \quad (8.1)$$

of the general equivalence group of all linear ordinary differential equations. We will find now all linear second-order equations (6.1) admitting the operator (8.1) with any *fixed* functions  $\phi(x)$  and  $\sigma(x)$ .

Taking the second prolongation of the operator (8.1),

$$X_1 = \phi \frac{\partial}{\partial x} + \sigma y \frac{\partial}{\partial y} + [\sigma' y + (\sigma - \phi') y'] \frac{\partial}{\partial y'} + [\sigma'' y + (2\sigma' - \phi'') y' + (\sigma - 2\phi') y''] \frac{\partial}{\partial y''},$$

and writing the invariance condition of Eq. (6.1),

$$X_1(y'' + f(x)y' + g(x)y) \Big|_{(6.1)} = 0,$$

we obtain:

$$(\phi f' + f\phi' + 2\sigma' - \phi'')y' + (\phi g' + 2\phi'g + f\sigma' + \sigma'')y = 0.$$

It follows:

$$\begin{aligned}\phi f' + f\phi' + 2\sigma' - \phi'' &= 0, \\ \phi g' + 2\phi'g + f\sigma' + \sigma'' &= 0.\end{aligned}\tag{8.2}$$

The first equation (8.2) is written

$$(\phi f)' = (\phi' - 2\sigma)'$$

and yields:

$$f(x) = \frac{1}{\phi} [A + \phi' - 2\sigma], \quad A = \text{const.}\tag{8.3}$$

Substituting this in the second equation (8.2), we obtain the following non-homogeneous linear first-order equation for determining  $g(x)$  :

$$\phi g' + 2\phi'g = -\sigma'' - \frac{\sigma'}{\phi} [A + \phi' - 2\sigma].\tag{8.4}$$

The homogeneous equation

$$\phi(x)g' + 2\phi'(x)g = 0$$

with a given function  $\phi(x)$  yields

$$g = \frac{C}{\phi^2(x)}.$$

By variation of the parameter  $C$ , we set

$$g = \frac{u(x)}{\phi^2(x)},$$

substitute it in Eq. (8.4) and obtain:

$$u' = -A\sigma' + 2\sigma\sigma' - \phi'\sigma' - \phi\sigma'' \equiv -(A\sigma)' + (\sigma^2)' - (\phi\sigma')',$$

whence

$$u = B - A\sigma + \sigma^2 - \phi\sigma', \quad B = \text{const.}$$

Therefore,

$$g = \frac{1}{\phi^2(x)} [B - A\sigma + \sigma^2 - \phi\sigma'].\tag{8.5}$$

Thus, we have arrived at the following result.

**Theorem 8.1.** The homogeneous linear second-order equations (6.1) admitting the operator (8.1) with any given functions  $\phi = \phi(x)$  and  $\sigma = \sigma(x)$  have the form

$$\phi^2 y'' + (A + \phi' - 2\sigma)\phi y' + (B - A\sigma + \sigma^2 - \phi\sigma')y = 0. \quad (8.6)$$

Now we use the homogeneity of Eq. (8.6) characterized by the generator

$$X_2 = y \frac{\partial}{\partial y}.$$

Namely, we look for the invariant solutions with respect to the linear combination  $X = X_1 + \lambda X_2$ :

$$X = \phi(x) \frac{\partial}{\partial x} + (\sigma(x) + \lambda)y \frac{\partial}{\partial y}, \quad \lambda = \text{const.},$$

and arrive at the following statement reducing the problem of integration of the differential equation (8.6) to solution of the quadratic equation, namely, the characteristic equation as in the case of equations with constant coefficients.

**Theorem 8.2.** Eq. (8.6) has the invariant solutions of the form

$$y = e^{\int \frac{\sigma(x) + \lambda}{\phi(x)} dx}, \quad (8.7)$$

where  $\lambda$  satisfies the *characteristic equation*

$$\lambda^2 + A\lambda + B = 0. \quad (8.8)$$

**Proof.** We solve the equation  $X(J) = 0$  for the invariants  $J(x, y)$ , i.e. integrate the equation

$$\frac{dy}{y} = \frac{\sigma(x) + \lambda}{\phi(x)} dx$$

and obtain the invariant

$$J = y e^{-\int \frac{\sigma(x) + \lambda}{\phi(x)} dx}.$$

Setting  $J = C$  and letting  $C = 1$  we obtain Eq. (8.7) for the invariant solutions. Thus, we have:

$$y = e^{\int \frac{\sigma(x) + \lambda}{\phi(x)} dx}, \quad y' = \frac{\sigma + \lambda}{\phi} e^{\int \frac{\sigma(x) + \lambda}{\phi(x)} dx} \quad (8.9)$$

$$y'' = \frac{1}{\phi^2} [(\sigma + \lambda)^2 - (\sigma + \lambda)\phi' + \phi\sigma'] e^{\int \frac{\sigma(x) + \lambda}{\phi(x)} dx}.$$

Substituting (8.9) in Eq. (8.6) we obtain Eq. (8.8), thus completing the proof.

## 8.2 Distinct roots of the characteristic equation

It is manifest that if the characteristic equation (8.8) has distinct real roots  $\lambda_1 \neq \lambda_2$ , the general solution to Eq. (8.6) is given by

$$y(x) = K_1 e^{\int \frac{\sigma(x)+\lambda_1}{\phi(x)} dx} + K_2 e^{\int \frac{\sigma(x)+\lambda_2}{\phi(x)} dx}, \quad K_1, K_2 = \text{const.} \quad (8.10)$$

In the case of complex roots,  $\lambda_1 = \gamma + i\theta$ ,  $\lambda_2 = \gamma - i\theta$ , the general solution to Eq. (8.6) is given by

$$y(x) = \left[ K_1 \cos \left( \theta \int \frac{dx}{\phi(x)} \right) + K_2 \sin \left( \theta \int \frac{dx}{\phi(x)} \right) \right] e^{\int \frac{\sigma(x)+\gamma}{\phi(x)} dx}. \quad (8.11)$$

## 8.3 The case of repeated roots

**Theorem 8.3.** If the characteristic equation (8.8) has equal roots  $\lambda_1 = \lambda_2$ , the general solution to Eq. (8.6) is given by

$$y = \left[ K_1 + K_2 \int \frac{dx}{\phi(x)} \right] e^{\int \frac{\sigma(x)+\lambda_1}{\phi(x)} dx}, \quad K_1, K_2 = \text{const.} \quad (8.12)$$

**Proof.** Let us new variables  $t$  and  $z$  defined by the linear first-order equations

$$X_1(t) \equiv \phi(x) \frac{\partial t}{\partial x} + \sigma(x) y \frac{\partial t}{\partial y} = 1, \quad X_2(t) \equiv y \frac{\partial t}{\partial y} = 0 \quad (8.13)$$

and

$$X_1(z) \equiv \phi(x) \frac{\partial z}{\partial x} + \sigma(x) y \frac{\partial z}{\partial y} = 0, \quad X_2(z) \equiv y \frac{\partial z}{\partial y} = z, \quad (8.14)$$

respectively. Eqs. (8.13) are easily solved and yield

$$t = \int \frac{dx}{\phi(x)}. \quad (8.15)$$

Integration of the second equation (8.14) with respect to  $y$  gives

$$z = v(x)y.$$

Substituting this in the first equation (8.14) we obtain

$$\phi(x) \frac{dv}{dx} + \sigma(x) v = 0, \quad \text{whence} \quad v = e^{-\int \frac{\sigma(x)}{\phi(x)} dx}.$$

Thus,

$$z = y e^{-\int \frac{\sigma(x)}{\phi(x)} dx}. \quad (8.16)$$

The passage to the new variables (8.15), (8.16) converts the operator  $X_1$  given by (8.1) to the translation generator without changing the form of the dilation generator  $X_2$ . In other words, upon introducing the new independent and dependent variables  $t$  and  $z$  given by (8.15) and (8.16), respectively, we arrive at the operators (6.3). Hence, in the new variables, Eq. (8.16) becomes an equation with constant coefficients. Invoking that the equations (8.6) and (6.2) have Eq. (8.8) as their common characteristic equation, we use Lemma 6.1 and write

$$z = (K_1 + K_2 t) e^{\lambda_1 t}.$$

Substituting this in Eq. (8.16) and replacing  $t$  and  $z$  by their expressions (8.15) and (8.16), respectively, and solving for  $y$ , we finally arrive at Eq. (8.12).

**Remark 8.1.** We can easily solve the non-homogeneous equation Eq. (8.6):

$$\phi^2 y'' + (A + \phi' - 2\sigma)\phi y' + (B - A\sigma + \sigma^2 - \phi\sigma')y = F(x). \quad (8.17)$$

Namely, we rewrite Eq. (8.6) in the form

$$y'' + a(x)y' + b(x)y = P(x)$$

and employ the representation of the general solution (see, e.g. [2], Section 3.3.4)

$$y = K_1 y_1(x) + K_2 y_2(x) - y_1(x) \int \frac{y_2(x)P(x)}{W(x)} dx + y_2(x) \int \frac{y_1(x)P(x)}{W(x)} dx \quad (8.18)$$

furnished by the method of variation of parameters. Here

$$W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x)$$

is the Wronskian of a fundamental system of solutions  $y_1(x)$ ,  $y_2(x)$  for the homogeneous equation

$$y'' + a(x)y' + b(x)y = 0.$$

## 9 Examples to Section 8

Euler's substitution (6.5) as well as the solutions (6.19), (7.5) and (7.12) are encapsulated in Eq. (8.7). We will consider now these and several other examples.

**Example 9.1.** Let us take  $\phi(x) = 1$ ,  $\sigma(x) = 0$ . Then Eqs. (8.6), (8.7) and (8.12) coincide with Eqs. (6.2), (6.5) and (6.9), respectively. Eq. (8.11) becomes (6.8).

**Example 9.2.** Let us take  $\phi(x) = x$ ,  $\sigma(x) = 0$ . Then Eq. (8.6) becomes Euler's equation written in the form (6.21), Eq. (8.7) yields Eq. (6.19) for invariant solutions, whereas Eq. (8.12) provides the general solution

$$y(x) = (K_1 + K_2 \ln x) x^{\lambda_1} \quad K_1, K_2 = \text{const.}, \quad (9.1)$$

to Euler's equation (6.21) whose characteristic equation (6.6) has equal roots. Eq. (8.11) leads to the following solution for complex roots  $\lambda_1 = \gamma + i\theta$ ,  $\lambda_2 = \gamma - i\theta$ :

$$y(x) = [K_1 \cos(\theta \ln x) + K_2 \sin(\theta \ln x)] x^\gamma. \quad (9.2)$$

**Example 9.3.** Let us take  $\phi(x) = x^\alpha$ ,  $\sigma(x) = 0$ . Then Eqs. (8.6) and (8.7) coincide with Eqs. (7.4) and (7.5), respectively, whereas Eq. (8.12) provides the general solution

$$y(x) = (K_1 + K_2 x^{1-\alpha}) e^{\frac{\lambda_1}{1-\alpha} x^{1-\alpha}}, \quad K_1, K_2 = \text{const.}, \quad (9.3)$$

to Eq. (7.4) whose characteristic equation (7.6) has equal roots  $\lambda_1 = \lambda_2$ . Eq. (9.3) extends the solution (7.8) to all equations (7.4) with the coefficients  $A, B, C$  satisfying the condition (6.12) of equal roots for the characteristic equation (7.6).

**Example 9.4.** Let us take  $\phi(x) = 1 + x^2$ ,  $\sigma(x) = x$ . Then Eq. (8.6) becomes

$$(1 + x^2)^2 y'' + (1 + x^2) A y' + (B - Ax - 1)y = 0. \quad (9.4)$$

Working out the integral in Eq. (8.7),

$$\int \frac{\sigma(x) + \lambda}{\phi(x)} dx = \int \frac{x}{1 + x^2} dx + \int \frac{\lambda}{1 + x^2} dx = \ln \sqrt{1 + x^2} + \lambda \arctan x,$$

we obtain the following expression for the invariant solutions:

$$y = \sqrt{1 + x^2} e^{\lambda \arctan x}, \quad (9.5)$$

where  $\lambda$  satisfies the characteristic equation (8.8):

$$\lambda^2 + A\lambda + B = 0. \quad (9.6)$$

If the characteristic equation (9.6) has distinct real roots,  $\lambda_1 \neq \lambda_2$ , the general solution to Eq. (9.4) is given by

$$y(x) = \sqrt{1+x^2} \left[ K_1 e^{\lambda_1 \arctan x} + K_2 e^{\lambda_2 \arctan x} \right]. \quad (9.7)$$

In the case of complex roots,  $\lambda_1 = \gamma + i\theta$ ,  $\lambda_2 = \gamma - i\theta$ , the general solution to Eq. (9.4) is given by

$$y(x) = [K_1 \cos(\theta \arctan x) + K_2 \sin(\theta \arctan x)] \sqrt{1+x^2} e^{\gamma \arctan x}. \quad (9.8)$$

Finally, if the characteristic equation (9.6) has equal roots  $\lambda_1 = \lambda_2$ , the general solution to Eq. (9.4) is given by

$$y(x) = (K_1 + K_2 \arctan x) \sqrt{1+x^2} e^{\lambda \arctan x}. \quad (9.9)$$

**Example 9.5.** Consider Eq. (9.4) with  $A = 0$ ,  $B = \omega^2$ . Then, according to Example 9.4, the equation

$$(1+x^2)^2 y'' + (\omega^2 - 1)y = 0 \quad (9.10)$$

has the following general solution:

$$y(x) = [K_1 \cos(\omega \arctan x) + K_2 \sin(\omega \arctan x)] \sqrt{1+x^2}. \quad (9.11)$$

**Example 9.6.** Let us solve the non-homogeneous equation

$$(1+x^2)^2 y'' + (\omega^2 - 1)y = F(x). \quad (9.12)$$

Example 9.5 provides the fundamental system of solutions

$$y_1 = \sqrt{1+x^2} \cos(\omega \arctan x), \quad y_2 = \sqrt{1+x^2} \sin(\omega \arctan x)$$

for the homogeneous equation (9.10). We have:

$$y_1' = \frac{1}{\sqrt{1+x^2}} [x \cos(\omega \arctan x) - \omega \sin(\omega \arctan x)],$$

$$y_2' = \frac{1}{\sqrt{1+x^2}} [x \sin(\omega \arctan x) + \omega \cos(\omega \arctan x)].$$

Hence the Wronskian is  $W[y_1, y_2] = y_1 y_2' - y_2 y_1' = \omega$ . Now we rewrite Eq. (9.12), in accordance with Remark 8.1, in the form

$$y'' + \frac{\omega^2 - 1}{(1 + x^2)^2} y = \frac{F(x)}{(1 + x^2)^2}, \quad (9.13)$$

use Eq. (8.18) and obtain the following general solution to Eq. (9.12):

$$\begin{aligned} y(x) = & \sqrt{1 + x^2} \left[ K_1 \cos(\omega \arctan x) + K_2 \sin(\omega \arctan x) \right. \\ & - \frac{1}{\omega} \cos(\omega \arctan x) \int \frac{F(x)}{(1 + x^2)^{3/2}} \sin(\omega \arctan x) dx \\ & \left. + \frac{1}{\omega} \sin(\omega \arctan x) \int \frac{F(x)}{(1 + x^2)^{3/2}} \cos(\omega \arctan x) dx \right]. \end{aligned} \quad (9.14)$$

## 10 Third-order equations

The previous results can be extended to higher-order linear ordinary differential equations. I will discuss here the third-order equations

$$y''' + f(x)y'' + g(x)y' + h(x)y = 0. \quad (10.1)$$

**Theorem 10.1.** The homogeneous linear third-order equations (10.1) admitting the operator (8.1) with any given  $\phi = \phi(x)$  and  $\sigma = \sigma(x)$  have the form

$$\begin{aligned} & \phi^3 y''' + [A + 3(\phi' - \sigma)] \phi^2 y'' \\ & + [B + A\phi' - 2A\sigma + \phi\phi'' + (\phi')^2 - 3(\phi\sigma)' + 3\sigma^2] \phi y' \\ & + [C - B\sigma + A\sigma^2 - A\phi\sigma' - \sigma^3 - \phi^2\sigma'' - \phi\phi'\sigma' + 3\phi\sigma\sigma'] y = 0. \end{aligned} \quad (10.2)$$

**Proof.** We take the third prolongation of the operator (8.1):

$$\begin{aligned} X_1 = & \phi \frac{\partial}{\partial x} + \sigma y \frac{\partial}{\partial y} + [\sigma' y + (\sigma - \phi') y'] \frac{\partial}{\partial y'} \\ & + [\sigma'' y + (2\sigma' - \phi'') y' + (\sigma - 2\phi') y''] \frac{\partial}{\partial y''} \\ & + [\sigma''' y + (3\sigma'' - \phi''') y' + 3(\sigma' - \phi'') y'' + (\sigma - 3\phi') y'''] \frac{\partial}{\partial y'''} \end{aligned}$$

and write the invariance condition of Eq. (10.1):

$$X_1(y''' + f(x)y'' + g(x)y' + h(x)y) \Big|_{(10.1)} = 0.$$

We annul the coefficients for  $y''$ ,  $y'$  and  $y$  of the left-hand side of the above equation and split it into the following three equations:

$$\phi f' + \phi' f + 3(\sigma' - \phi'') = 0, \quad (10.3)$$

$$\phi g' + 2\phi' g + (2\sigma' - \phi'')f - \phi''' + 3\sigma'' = 0, \quad (10.4)$$

$$\phi h' + 3\phi' h + \sigma'' f + \sigma' g + \sigma''' = 0. \quad (10.5)$$

Eq. (10.3) is written

$$(\phi f)' = 3(\phi' - \sigma)'$$

and yields:

$$f = \frac{1}{\phi} [A + 3(\phi' - \sigma)], \quad A = \text{const.} \quad (10.6)$$

We substitute (10.6) in Eq. (10.4), integrate the resulting non-homogeneous linear first-order equation for  $g$  and obtain:

$$g = \frac{1}{\phi^2} [B + A\phi' - 2A\sigma + \phi\phi'' + (\phi')^2 - 3(\phi\sigma)' + 3\sigma^2], \quad (10.7)$$

where  $B$  is an arbitrary constant. Now we substitute (10.6), (10.7) in Eq. (10.5), integrate the resulting first-order equation for  $h$  and obtain:

$$h = \frac{1}{\phi^3} [C - B\sigma + A\sigma^2 - A\phi\sigma' - \sigma^3 - \phi^2\sigma'' - \phi\phi'\sigma' + 3\phi\sigma\sigma'], \quad (10.8)$$

where  $C$  is an arbitrary constant. Finally, substituting (10.6), (10.7) and (10.8) in Eq. (10.1), we arrive at Eq. (10.2).

**Theorem 10.2.** Eq. (10.2) has the invariant solutions of the form (8.7),

$$y = e^{\int \frac{\sigma(x) + \lambda}{\phi(x)} dx}, \quad (8.7)$$

where  $\lambda$  satisfies the *characteristic equation*

$$\lambda^3 + A\lambda^2 + B\lambda + C = 0. \quad (10.9)$$

**Proof.** Adding to Eqs. (8.9) the expression for the third derivative  $y'''$  and substituting in Eq. (10.2) we obtain Eq. (10.9).

# Chapter 3

## Using connection between Riccati and second-order linear equations

### 11 Introduction

Recall that the Riccati equation

$$y' = P(x) + Q(x)y + R(x)y^2, \quad R(x) \neq 0, \quad (11.1)$$

is mapped by the substitution

$$y = -\frac{1}{R(x)} \frac{u'}{u} \quad (11.2)$$

to the linear second-order equation

$$u'' + f(x)u' + g(x)u = 0 \quad (11.3)$$

with the coefficients

$$f(x) = -\left[Q(x) + \frac{R'(x)}{R(x)}\right], \quad g(x) = P(x)R(x). \quad (11.4)$$

Indeed, we have:

$$y' = -\frac{1}{R} \frac{u''}{u} + \frac{R'}{R^2} \frac{u'}{u} + \frac{1}{R} \frac{u'^2}{u^2}$$

and

$$P + Qy + Ry^2 = P - \frac{Q}{R} \frac{u'}{u} + \frac{1}{R} \frac{u'^2}{u^2}.$$

Substituting these expressions in Eq. (11.1) and multiplying by  $-Ru$  we obtain the equation

$$u'' - \frac{R'}{R} u' = Qu' - PRu,$$

i.e. Eq. (11.3) with the coefficients (11.4).

## 12 From Riccati to second-order equations

### 12.1 Application to Equation (2.9)

Applying Eqs. (11.3)-(11.4) to Eq. (2.9),

$$y' = Q(x)y + R(x)y^2, \quad (2.9)$$

we obtain the following second-order linear equation:

$$u'' = \left(Q + \frac{R'}{R}\right)u'. \quad (12.1)$$

The integration yields:

$$\ln u' = \int \left(Q + \frac{R'}{R}\right) dx + \ln C_1 = \int Q dx + \ln R + \ln C_1.$$

Hence

$$u' = C_1 R(x) e^{\int Q(x) dx} \quad (12.2)$$

and

$$u = C_1 \int R(x) e^{\int Q(x) dx} dx + C_2. \quad (12.3)$$

Substituting (12.2) and (12.3) in Eq. (11.2) and denoting  $C = C_2/C_1$ , we arrive at the solution (3.6) to Eq. (2.9):

$$y = -\frac{e^{\int Q(x) dx}}{C + \int R(x) e^{\int Q(x) dx} dx}. \quad (3.6)$$

### 12.2 Application to Equation (2.10)

The following examples clarify how to use the linearizable equations (2.10),

$$y' = P(x) + Q(x)y + k[Q(x) - kP(x)]y^2, \quad k = \text{const.}, \quad (2.10)$$

for integrating the corresponding second-order linear equations (11.3).

**Example 12.1.** If we apply Eqs. (11.3)-(11.4) to Eq. (4.8) from Example 4.1,

$$y' = x + 2xy + xy^2,$$

we obtain the following second-order linear equation:

$$u'' - \left(2x + \frac{1}{x}\right)u' + x^2u = 0. \quad (12.4)$$

Let us integrate this equation. Writing Eq. (11.2) in the form

$$\frac{u'}{u} = -R(x)y,$$

substituting here  $R(x) = x$  and the expression (4.9) for  $y$ , we get

$$\frac{u'}{u} = -x \frac{1 + C + x^2}{1 - C - x^2}.$$

Writing this equation in the form

$$\frac{d \ln u}{dx} = x + \frac{2x}{x^2 + C - 1}$$

and integrating we obtain the following general solution to Eq. (12.4):

$$u = K(x^2 + C - 1)e^{x^2/2}, \quad C, K = \text{const.} \quad (12.5)$$

**Example 12.2.** If we apply Eqs. (11.3)-(11.4) to Eq. (4.10) from Example 4.2,

$$y' = x^2 + (x + x^2)y + \frac{1}{4}(2x + x^2)y^2,$$

we obtain the following second-order linear equation:

$$u'' - (1 + x) \left( x + \frac{2}{2x + x^2} \right) u' + \frac{1}{4}x^2(2x + x^2)u = 0. \quad (12.6)$$

Let us integrate this equation. Writing Eq. (11.2) in the form

$$\frac{u'}{u} = -R(x)y,$$

substituting here

$$R(x) = \frac{1}{4}(2x + x^2)$$

and the expression (4.11) for  $y$ , we get

$$\frac{u'}{u} = -\frac{1}{2}(2x + x^2) \frac{1 + \left(C + \int (x + x^2)e^{-x^2/2} dx\right) e^{x^2/2}}{1 - \left(C + \int (x + x^2)e^{-x^2/2} dx\right) e^{x^2/2}}.$$

The integration yields  $\ln |u| = \ln |K| + \phi(x)$ , where

$$\phi(x) = -\frac{1}{2} \int (2x + x^2) \frac{1 + \left(C + \int (x + x^2)e^{-x^2/2} dx\right) e^{x^2/2}}{1 - \left(C + \int (x + x^2)e^{-x^2/2} dx\right) e^{x^2/2}} dx.$$

Hence, the general solution to Eq. (12.6) has the form

$$u = K e^{\phi(x)}, \quad (12.7)$$

where  $\phi(x)$  is given above and  $K$  is an arbitrary constant.

Applying Eqs. (11.3)-(11.4) to Eq. (2.10) and using the solution procedure for Eq. (2.10) described in Section 4, we obtain the following general result.

**Theorem 12.1.** The general solution of the second-order linear equation

$$u'' - \left[ Q(x)x + \frac{Q'(x) - kP'(x)}{Q(x) - kP(x)} \right] u' + k[P(x)Q(x) - kP^2(x)]u = 0 \quad (12.8)$$

with an arbitrary constant  $k$  and two arbitrary functions  $P(x)$  and  $Q(x)$  can be obtained by quadratures.

## 13 From second-order to Riccati equations

It is manifest from Eqs. (11.4) that *two* coefficients  $f(x)$  and  $g(x)$  of a given second-order equation (11.3) do not uniquely determine *three* coefficients  $P(x), Q(x), R(x)$  of the corresponding Riccati equation (11.1). Namely, if we know solutions of an equation (11.3), we can solve by using the formula (11.2) an infinite set of the Riccati equations

$$y' = R(x)y^2 - \left[ f(x) + \frac{R'(x)}{R(x)} \right] y + \frac{g(x)}{R(x)} \quad (13.1)$$

with an arbitrary function  $R(x) \neq 0$ .

**Example 13.1.** Consider the following equation with constant coefficients:

$$u'' + u = 0. \quad (13.2)$$

Here  $f = 0$ ,  $g = 1$ . Hence, the corresponding Riccati equation (13.1) has the form

$$y' = R(x) y^2 - \frac{R'(x)}{R(x)} y + \frac{1}{R(x)}. \quad (13.3)$$

Substituting the general solution

$$u = C_1 \cos x + C_2 \sin x$$

of Eq. (13.2) in (11.2) we obtain the following solution to Eq. (13.3):

$$y = \frac{1}{R(x)} \frac{C_1 \sin x - C_2 \cos x}{C_1 \cos x + C_2 \sin x}.$$

If  $C_2 \neq 0$  we denote  $K = C_2/C_1$  and write the solution in the form

$$y = \frac{1}{R(x)} \frac{\sin x - K \cos x}{\cos x + K \sin x}$$

or, upon dividing the numerator and denominator by  $\cos x$ ,

$$y = \frac{1}{R(x)} \frac{\operatorname{tg} x - K}{1 + K \operatorname{tg} x}, \quad K = \text{const.} \quad (13.4)$$

If  $C_2 = 0$  the solution becomes

$$y = -\frac{\operatorname{ctg} x}{R(x)}$$

which can be obtained from (13.4) by letting  $K \rightarrow \infty$ . Thus, the general solution to the Riccati equation (13.5) is given by (13.4) where  $-\infty \leq K \leq +\infty$ .

In particular, taking in (13.5), (13.4)  $R(x) = e^x$  we conclude that the equation

$$y' = e^x y^2 - y + e^{-x} \quad (13.5)$$

has the general solution

$$y = \frac{\operatorname{tg} x - K}{1 + K \operatorname{tg} x} e^{-x}, \quad -\infty \leq K \leq +\infty. \quad (13.6)$$

Sometimes it is convenient to use a restricted correspondence between second-order and Riccati equations by writing Eqs. (11.3)-(11.4) corresponding to the Riccati equation (11.1) in the following form:

$$R(x) u'' - [R'(x) + Q(x)R(x)] u' + P(x)R^2(x) u = 0. \quad (13.7)$$

**Example 13.2.** Consider Euler's equation written in the form (6.21):

$$x^2 u'' + (A + 1) x u' + B u = 0, \quad A, B = \text{const.} \quad (13.8)$$

Comparing the equations (13.7) and (13.8) we take  $R(x) = x^2$  and obtain

$$Q(x) = -\frac{A+3}{x} \quad P(x) = \frac{B}{x^4}.$$

Thus, we have arrived at the following Riccati equation:

$$y' = x^2 y^2 - \frac{A+3}{x} y + \frac{B}{x^4}. \quad (13.9)$$

We know that the solutions of Eq. (13.8) have the form  $u = x^\lambda$ . Substitution in (11.2) yields the following form of solutions to Eq. (13.9):

$$y = -\frac{\lambda}{x^3}. \quad (13.10)$$

Substituting (13.10) in Eq. (13.9) we obtain again the characteristic equation (6.6):

$$\lambda^2 + A\lambda + B = 0. \quad (13.11)$$

However, Eqs. (13.10), (13.11) provide only two particular solutions to Eq. (13.9). In order to find the general solution of the *nonlinear* equation (13.9), we have to construct the general solution  $u(x)$  of the *linear* equation (13.8) using the *superposition principle* and substitute  $u = u(x)$  in (11.2).

Using the results of Section 8 on integrability of Eq. (8.6),

$$\phi^2 y'' + (A + \phi' - 2\sigma)\phi y' + (B - A\sigma + \sigma^2 - \phi\sigma')y = 0,$$

we can formulate the following general result.

**Theorem 13.1.** The Riccati equation

$$y' = R(x) y^2 - \left[ \frac{A + \phi' - 2\sigma}{\phi} + \frac{R'}{R} \right] y + \frac{B - A\sigma + \sigma^2 - \phi\sigma'}{R\phi^2} \quad (13.12)$$

with three arbitrary functions  $R(x)$ ,  $\phi(x)$ ,  $\sigma(x)$  and two arbitrary constants  $A$ ,  $B$  is integrable by quadratures.

## 14 Application to Ermakov's equation

The above results on integration of linear equations

$$u'' + a(x)u' + b(x)u = 0 \quad (14.1)$$

can be combined with Ermakov's method for solving nonlinear equations of the

following form (see Editor's preface to Ermakov's paper in this volume):

$$u'' + a(x)u' + b(x)u = \frac{\alpha}{u^3} e^{-2 \int a(x) dx}, \quad \alpha = \text{const.} \quad (14.2)$$

**Example 14.1.** Using the solution (12.5) of Eq. (12.4) and applying Ermakov's method one can solve the following nonlinear equation:

$$u'' - \left(2x + \frac{1}{x}\right)u' + x^2u = \alpha x^2 e^{2x^2} u^{-3}. \quad (14.3)$$

**Example 14.2.** The nonlinear equation (14.2) associated with the integrable equation (8.6) has the form

$$\phi^2 u'' + (A + \phi' - 2\sigma)\phi u' + (B - A\sigma + \sigma^2 - \phi\sigma')u = \frac{\alpha}{u^3} e^{2 \int (2\sigma - A)/\phi dx}, \quad (14.4)$$

where  $\phi$  and  $\sigma$  are arbitrary functions of  $x$ , and  $A$  is an arbitrary constant. Eq. (14.4) is integrable by quadratures.

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